Radiation from an Oscillating Magnetic
Dipole in a Streaming Plasma (1)

John E. Bergeson

Department of Physics and Astronomy, University of Iowa
Iowa City, Iowa, 52240, U.S.A.

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Abstract

The electromagnetic field of an oscillating magnetic dipole is calculated, assuming that the dipole is immersed in a cold, streaming plasma. The amplitude of the magnetic dipole moment, assumed known, is taken to be sufficiently weak that the linearized cold plasma equations may be used to describe the response of the plasma.

The resulting field of the dipole is rather different from the field that would result if the plasma were not streaming. In particular, a longitudinal electrostatic field appears as a consequence of the plasma's motion. The far field of the dipole is such that the Poynting vector is not purely radial, but is tilted against the direction of the zeroth-order plasma flow.

An outward flow of mechanical energy is associated with the electrostatic field. However, the mechanical energy flow is negligible for streaming velocities small compared with the velocity of light. The force necessary to hold the dipole in place is also calculated. This force vanishes when the dipole axis is parallel to the streaming direction, as does the longitudinal electric field.

1. Introduction

Electromagnetic fields produced by given time-varying current and charge distributions in free space have been the subjects of many calculations, almost from the beginning of electromagnetic theory. Many problems of current interest are complicated by the fact that the fields may be interacting with a plasma or ionized gas. Several problems have been solved in various approximations in recent years for given distributions of charge and current in the presence of a plasma [see, for example, Cohen (1961, 62)]. But many gaps remain in our qualitative understanding of the fields to be expected in particular types of plasma situations.

In particular, effects associated with plasma streaming have been investigated relatively little. The calculations which have been done have, to a considerable extent, been concerned with plasmas which, to lowest order, are assumed to be stationary. It is to be expected that some insight into the effects of plasma streaming can be gained by seeking specific solvable problems concerned with radiation into streaming plasmas.

It is the purpose of this paper to study the effect of a cold, streaming plasma on the electric and magnetic fields of an oscillating magnetic point dipole. The plasma is unbounded and it streams across the dipole in an arbitrary direction with respect to the dipole's orientation. We assume that the fields of the dipole are sufficiently weak that they impart only a small perturbation to the streaming motion of the plasma. We also assume that the physical dimensions of the dipole are so small that it does not mechanically obstruct the plasma flow; i.e., we idealize the dipole as a point.

Lee and Papas (1965) have considered a similar problem for the oscillating electric dipole. After obtaining an integral representation for the potential four-vector in the rest frame of the dipole, they use the transformation properties of the plasma's dielectric constant to calculate the appropriate Green's function. For streaming velocities small compared with the velocity of light, they conclude that the far zone electromagnetic field is not entirely transverse. As a result, they show that the Poynting vector associated with the oscillating electric dipole is tilted against the direction of plasma flow.

We find qualitatively similar results for the oscillating magnetic dipole, although our results differ considerably in detail. Moreover, we approach the problem from a different point-of-view. Our formalism is based on a set of linearized, covariant, cold plasma equations wherein we treat the magnetic dipole as a small, external current "source" in the sense proposed (for example) by Cohen (1961). That is, we represent the dipole by a miniature current loop which weakly perturbs the streaming plasma. We assume that we may prescribe the current in the loop at will. The physical reason for the difference between our results and those of Lee and Papas is that their electric dipole possesses, in effect, an oscillating source charge density, whereas our magnetic dipole is a pure divergenceless current source.

2. The Magnetic Dipole

2.1 Statement of the Problem

A cold, collisionless plasma streams across a circular loop of oscillating current (figure 1). The orientation of the flow vector $\vec{v}_o$ relative to the plane of the loop is arbitrary. We treat the loop as an externally-fixed current source which is unaffected by the plasma. Later, we shall allow the area of the current loop to become vanishingly small and its current to become infinitely large in such a way that we recover a point magnetic dipole.

The current source generates disturbances in the plasma. However, we assume that the source is sufficiently weak and that the disturbances are sufficiently small that linearized cold plasma equations are applicable. We seek analytical expressions for the electric and magnetic fields of the oscillating current loop in the presence of a streaming plasma.

2.2 Solution in Wave-Vector, Frequency Space when $\vec{j}\ \mathrm{source}_{\parallel}\ (\vec{k},\omega) = 0$

The fields in the plasma are governed by the cold plasma equations:(2)

Maxwell Equations

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1a}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\left[\left(\sum_i e_i n_i \vec{v}_i\right) + \vec{j}_s\right] \tag{1b}$$

$$\nabla \cdot \vec{E} = 4\pi \left[\sum_i e_i n_i + \rho_s\right] \tag{1c}$$

$$\nabla \cdot \vec{B} = 0 \tag{1d}$$

Equation of Continuity

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{v}_i) = 0 \tag{1e}$$

Equation of Motion

$$\frac{\partial \vec{v}_i}{\partial t} + \vec{v}_i \cdot \nabla \vec{v}_i = \frac{e_i}{m_i \gamma_i}\left[\vec{E} + \frac{1}{c}(\vec{v}_i \times \vec{B}) - \frac{\vec{v}_i \vec{v}_i}{c^2} \cdot \vec{E}\right] \tag{1f}$$

2) Gaussian units are used throughout this paper since we treat the plasma "microscopically" and write Maxwell's equations in terms of the fundamental fields. Moreover, much of the relevant plasma literature is written in Gaussian units.

In the equations above, $\vec{j}_s$ and ρ_s are the externally-introduced "source" current and "source" charge densities, respectively; n_i and $\vec{v}_i$ are the number density and velocity, respectively, of the ith species of charge, and γ_i is the relativistic factor

$$\gamma_i = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}} \quad .$$

The equation of continuity and Maxwell's equations are covariant. The equation of motion is not covariant, although it is correct up to terms of $O\ (v_i^4/c^4)$, and can be derived from the covariant equation of motion by neglecting fourth order terms in $\frac{v_i}{c}$. We have neglected a pressure term in the equation of motion, assuming that the plasma thermal velocities are small enough to be negligible.

We may linearize equations (1) by setting $\vec{E} = \vec{E}^{(1)}$, $\vec{B} = \vec{B}^{(1)}$, $n_i = n_{oi} + n_i^{(1)}$ and $\vec{v}_i = \vec{v}_o + \vec{v}_i^{(1)}$, where n_{oi} is the equilibrium number density of the ith plasma component (measured in the system with velocity $\vec{v}_o$) and $\vec{v}_o$ is the unperturbed streaming velocity common to all components. The superscript (1) signifies a first order perturbation. We assume that there are no zeroth-order electric or magnetic fields in the streaming plasma. We also assume that $\Sigma_i n_{oi} e_i = 0$, i.e. that there is no zeroth-order charge density, and that $\Sigma_i n_{oi} e_i \vec{v}_o = 0$, i.e. that there is no zeroth-order current density.

After Fourier analysis of the perturbation quantities in space and time, the linearized Maxwell equations may be combined to yield the wave equation

$$(\omega^2 - c^2 k^2)\,\vec{E}^{(1)} + c^2\,\vec{k}\vec{k}\cdot\vec{E}^{(1)} = -4\pi\, i\omega \sum_i e_i \left[n_o \vec{v}_i^{(1)} + n_i^{(1)}\,\vec{v}_o\right] - 4\pi\, i\omega\,\vec{j}_s\,, \tag{2}$$

where the Fourier transform of the electric field, for example, is given by

$$\vec{E}^{(1)}(\vec{k},\omega) = \frac{1}{(2\pi)^4}\int d\vec{x}\int dt\;\vec{E}^{(1)}(\vec{x},t)\,e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$$

Similarly, the Fourier transformed equation of continuity and equation of motion may be solved for $n_i^{(1)}$ and $\vec{v}_i^{(1)}$ in terms of $\vec{E}^{(1)}$. The result is

$$\vec{v}_i^{(1)} = - \frac{e_i}{m_i \gamma_o (i\omega)} \left\{ \overleftrightarrow{I} + \frac{\vec{k}\,\vec{v}_o}{\omega - \vec{k}\cdot\vec{v}_o} - \frac{\omega\,\vec{v}_o\vec{v}_o}{c^2(\omega - \vec{k}\cdot\vec{v}_o)} \right\} \cdot \vec{E}^{(1)} \tag{3a}$$

$$n_i^{(1)} = - \frac{n_{o_i} e_i}{m_i \gamma_o (i\omega)} \frac{\vec{k}}{(\omega - \vec{k}\cdot\vec{v}_o)} \cdot \left\{ \overleftrightarrow{I} + \frac{\vec{k}\,\vec{v}_o}{(\omega - \vec{k}\cdot\vec{v}_o)} - \frac{\omega\,\vec{v}_o\vec{v}_o}{c^2(\omega - \vec{k}\cdot\vec{v}_o)} \right\} \cdot \vec{E}^{(1)} \tag{3b}$$

[It will generally be obvious from the context whether the arguments of the perturbation variables are $(\vec{x},t)$ or $(\vec{k},\omega)$] In deriving (3a) and (3b), we have retained the $\frac{\vec{v}_o}{c} \times \vec{B}^{(1)}$ term in the equation of motion and have eliminated $\vec{B}^{(1)}$ by using (1a). The result of substituting (3a) and (3b) in (2) is

$$(\omega^2 - c^2 k^2)\,\vec{E}^{(1)} + c^2\,\vec{k}\vec{k}\cdot\vec{E}^{(1)}$$

$$- \frac{\sum_i \omega_{p_i}^2}{\gamma_o} \left[\overleftrightarrow{I} + \frac{(\vec{k}\,\vec{v}_o + \vec{v}_o\,\vec{k})}{(\omega - \vec{k}\cdot\vec{v}_o)} \right.$$

$$\left. + \frac{(k^2 - \omega^2/c^2)\,(\vec{v}_o\,\vec{v}_o)}{(\omega - \vec{k}\cdot\vec{v}_o)^2} \right] \cdot \vec{E}^{(1)}$$

$$= - 4\pi\, i\,\omega\,\vec{j}_s \,. \tag{4}$$

We wish to solve equation (4) for $\vec{E}^{(1)}(\vec{k}, \omega)$; we outline below a procedure for accomplishing the solution.

1) Resolve (4) into components which are transverse ($\perp$) and parallel ($\parallel$) to the wave vector $\vec{k}$. One of these components will contain only $\vec{j}_{s_\perp}$. The other will contain only $\vec{j}_{s_\parallel}$, which will turn out to be zero for the magnetic dipole.

2) Solve the transverse component for $\vec{v}_o \cdot \vec{E}_\perp^{(1)}$ as a function of $E_\parallel^{(1)}$.

3) Solve the parallel component for $E_\parallel^{(1)}$ in terms of $\vec{v}_o \cdot \vec{E}_\perp^{(1)}$.

4) Solve simultaneously for $\vec{v}_o \cdot \vec{E}_\perp^{(1)}$ and $E_\parallel^{(1)}$ and then solve the transverse component for $\vec{E}_\perp^{(1)}$.

The results for $\vec{j}_{s_\parallel} = 0$ are

$$\vec{E}_\perp^{(1)}(\vec{k},\omega) = -\frac{4\pi i \omega}{\left[\omega^2 - c^2 k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]}\left\{\vec{j}_{s_\perp} - \frac{\sum_i \omega_{p_i}^2 \left(\vec{j}_{s_\perp}\cdot\vec{v}_o\right)\vec{v}_{o_\perp}}{\gamma_o c^2\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]}\right\} \tag{5}$$

$$\vec{E}_\parallel^{(1)}(\vec{k},\omega) = \frac{-4\pi i\left(\sum_i \omega_{p_i}^2\right)\left[1 - \frac{\omega}{c^2}\left(\frac{\vec{k}\cdot\vec{v}_o}{k^2}\right)\right]\vec{k}\left[\vec{j}_{s_\perp}\cdot\vec{v}_o\right]}{\gamma_o\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]} \tag{6}$$

2.3 The Current Transform $\vec{j}_{source\,\perp}(\vec{k}, \omega)$ for the Magnetic Dipole

In order to evaluate (5) and (6) for the oscillating magnetic dipole, we require an explicit expression for the Fourier transform of the current loop. Let the radius of the loop be a_o and let the axis of the loop lie along the z axis of an (r, φ, z) coordinate system (figure 1). For a loop current $J_o \cos \omega_o t$, the current density is

$$\vec{j}_s(\vec{x},t) = J_o\, \hat{e}_\varphi\, \delta\,(r - a_o)\, \delta\,(z) \cos \omega_o t,$$

where ω_o is the external (constant) driving frequency. The Fourier transform of this expression is

$$\vec{j}_s(\vec{k},\omega) = \frac{ic}{16\pi^3}\, [\vec{k} \times \vec{\mu}_o]\, [\,\delta(\omega - \omega_o) + \delta(\omega + \omega_o)] \quad , \tag{7}$$

where $\vec{\mu}_o$ is normal to the plane of the current loop with magnitude

$$\left|\vec{\mu}_o\right| = \lim_{\substack{J_o \to \infty \\ a_o \to 0 \\ J_o a_o^2 = \text{constant}}} \frac{\pi J_o a_o^2}{c} .$$

The current transform (7) for the magnetic dipole is purely <u>transverse</u> to the wave vector $\vec{k}$.

2.4 The Electric Field When $\frac{|\vec{v}_o|}{c} \ll 1$

Equations (5) and (6) give the transverse and longitudinal components of the electric field in $(\vec{k}, \omega)$ space. In principle the <u>inverse</u> transforms of these equations specify the electric field vector in $(\vec{x}, t)$ space. In practice, however, we find that the $\vec{k}$ integration is difficult due to the factor

$$\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3} \right]$$

in the denominators. Separating the denominators into partial fractions circumvents this difficulty for small streaming velocities $\left(\frac{v_o}{c} << 1 \right)$. Let

$$\frac{1}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]\left[(\omega^2 - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]}$$

$$= \frac{Mk + N}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]} + \frac{Pk + Q}{\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]}, \tag{8}$$

where M,N,P and Q are unknown coefficients. It can be shown that if we disregard all terms of order $\left(\frac{v_o}{c}\right)^2$ and above, the coefficients P and Q in (8) vanish and we have

$$\frac{1}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]}$$

$$\simeq \frac{2\omega\,(\vec{k}\cdot\vec{v}_o) + \left(\omega^2 - \sum_i \omega_{p_i}^2\right)}{\left[\omega^2 - \sum_i \omega_{p_i}^2\right]^2\left[\omega^2 - c^2k^2 - \sum_i \omega_{p_i}^2\right]} \tag{9}$$

But (6) and the "distortion" term in (5) are already at least of order $\frac{v_o}{c}$. Therefore, if we again disregard terms of order $\left(\frac{v_o}{c}\right)^2$, we find that

$$\vec{E}_{\perp}^{(1)}(\vec{x},t) \simeq -(4\pi i) \int d\omega \int d\vec{k} \; \frac{\omega \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \vec{j}_{s_\perp}}{\left[\omega^2 - c^2k^2 - \sum_i \omega_{p_i}^2\right]} \tag{10}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) \simeq -(4\pi i) \left(\sum_i \omega_{p_i}^2\right) \int d\omega \int d\vec{k} \; \frac{\vec{k} \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} (\vec{j}_{s_\perp} \cdot \vec{v}_o)}{\left(\omega^2 - \sum_i \omega_{p_i}^2\right) \left[\omega^2 - c^2k^2 - \sum_i \omega_{p_i}^2\right]} \tag{11}$$

which are correct up to terms of order $\left(\frac{v_o}{c}\right)^2$. Using (7) for the current transform and removing the $\vec{k}$ operators from the integrand $(k \rightarrow -i\nabla)$, we may write (10) and (11) as

$$\vec{E}_{\perp}^{(1)}(\vec{x},t) = \frac{i}{4\pi^2 c} \nabla \times \left\{ \vec{\mu}_o \int d\omega \int d\vec{k} \; \frac{\omega\,[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}}{\left[k^2 - \frac{1}{c^2}\left(\omega^2 - \sum_i \omega_{p_i}^2\right)\right]} \right\} \tag{12}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) = \frac{\left(\sum_i \omega_{p_i}^2\right)}{4\pi^2} \nabla \left\{ \frac{\vec{v}_o}{c} \cdot \left[\nabla \times \left(\vec{\mu}_o \int d\omega \int d\vec{k} \; \frac{[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}}{\left[\omega^2 - \sum_i \omega_{p_i}^2\right]\left[k^2 - \frac{1}{c^2}\left(\omega^2 - \sum_i \omega_{p_i}^2\right)\right]}\right)\right]\right\} \tag{13}$$

The transverse electric field is undistorted for small streaming velocities; one would obtain the same expression for an oscillating magnetic dipole in a <u>stationary</u> plasma. On the other hand, the longitudinal electric field is new. It is directly proportional to the plasma's streaming velocity $\vec{v}_o$ and appears despite the fact that the source current is purely transverse.

We may now perform the $\vec{k}$ and ω integrations, using complex contour integration and the residue theorem in evaluating the integral over $d\vec{k}$. However, since the poles of the integrand lie on the real axis, the $\vec{k}$ integral, as it stands, is not well defined. We can remove the ambiguity in one of two ways: (i) the requirement can be imposed that only outgoing waves be present in the result; or, (ii) a small collisional damping term $-\nu_i \vec{v}_i^{\,(1)}$ can be added to the right hand side of (1f), where ν_i is the (assumed constant) collision frequency for the ith species. We choose the latter procedure. Which we use makes little difference, since we are restricting ourselves to the case where ω_o, $\omega_{p_i} \gg$ all ν_i, so that the collision frequencies do not appear in the eventual answers. The results of the $\vec{k}$ and ω integrations, for

$$|\omega_o| > \left(\sum_i \omega_{p_i}^2 \right)^{1/2}$$

are

$$\vec{E}_{\perp}^{(1)}(\vec{x},t) = -\frac{\omega_o}{c}\nabla \times \left\{ \frac{\vec{\mu}_o}{\left|\vec{x}\right|} \sin\left(a\left|\vec{x}\right| - \omega_o t\right) \right\} \tag{14}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) = \frac{\sum_i \omega_{p_i}^2}{\left(\omega_o^2 - \sum_i \omega_{p_i}^2\right)} \left\{ \nabla \left[\frac{\vec{v}_o}{c} \cdot \left(\nabla \times \vec{\mu}_o \frac{\cos\left(a\left|\vec{x}\right| - \omega_o t\right)}{\left|\vec{x}\right|} \right) \right] \right\} \tag{15}$$

$$\text{where} \quad a = \frac{\omega_o}{c}\left[1 - \frac{\sum_i \omega_{p_i}^2}{\omega_o^2}\right]^{\frac{1}{2}}$$

The expressions above are independent of any specific set of spatial coordinates. We now choose the coordinate configuration in figure 2. Let the y axis of an (xyz) Cartesian system point in the direction of the streaming velocity $\vec{v}_o$; the xz plane is normal to the direction of flow. We place the dipole at the origin of the coordinate system and let the plasma stream past. The polar angle θ_m, measured from the z axis, and the azimuthal angle φ_m, measured from the x axis, define the angular orientation of the dipole. The spherical coordinates (r, θ, φ) specify the position of an observer relative to the origin. In terms of the parameters and coordinates defined above, the components of the transverse electric field are

$$E_{\perp r} = 0$$

$$E_{\perp \theta} = -\left(\frac{\mu_o \omega_o}{c}\right) \sin\theta_m \sin(\varphi - \varphi_m)\left[\frac{a\cos(ar-\omega_o t)}{r} - \frac{\sin(ar-\omega_o t)}{r^2}\right]$$

$$E_{\perp \varphi} = \left(\frac{\mu_o \omega_o}{c}\right)\left\{[\sin\theta \cos\theta_m - \cos\theta \sin\theta_m \cos(\varphi - \varphi_m)] \cdot \left[\frac{a\cos(ar-\omega_o t)}{r} - \frac{\sin(ar-\omega_o t)}{r^2}\right]\right\} . \quad (16)$$

The components of the longitudinal electric field are

$$E_{\parallel r} = \frac{\mu_o \left(\sum_i \omega_{p_i}^2\right)}{\left(\omega_o^2 - \sum_i \omega_{p_i}^2\right)} \left(\frac{v_o}{c}\right) [\sin\theta \cos\varphi \cos\theta_m - \cos\theta \sin\theta_m \cos\varphi_m] \cdot$$

$$\left[\frac{a^2 \cos(ar-\omega_o t)}{r} - \frac{2a \sin(ar-\omega_o t)}{r^2} - \frac{2 \cos(ar-\omega_o t)}{r^3}\right]$$

$$E_{\parallel \theta} = \frac{\mu_o \left(\sum_i \omega_{p_i}^2\right)}{\left(\omega_o^2 - \sum_i \omega_{p_i}^2\right)} \left(\frac{v_o}{c}\right) [\cos\theta \cos\varphi \cos\theta_m + \sin\theta \sin\theta_m \cos\varphi_m] \cdot$$

$$\left[\frac{a \sin(ar-\omega_o t)}{r^2} + \frac{\cos(ar-\omega_o t)}{r^3}\right]$$

$$E_{\parallel \varphi} = -\frac{\mu_o \left(\sum_i \omega_{p_i}^2\right)}{\left(\omega_o^2 - \sum_i \omega_{p_i}^2\right)} \left(\frac{v_o}{c}\right) \sin\varphi \cos\theta_m$$

$$\left[\frac{a \sin(ar-\omega_o t)}{r^2} + \frac{\cos(ar-\omega_o t)}{r^3}\right]. \qquad (17)$$

2.5 The Magnetic Field $\vec{B}(\vec{x},t)$ when $\frac{|\vec{v}_o|}{c} << 1$

We use the Maxwell equation

$$\nabla \times \vec{E}^{(1)} = -\frac{1}{c}\frac{\partial \vec{B}^{(1)}}{\partial t}$$

to find the magnetic field $\vec{B}^{(1)}$ in the plasma; there is no magnetic field associated with $\vec{E}_{\parallel}^{(1)}$, which is derivable from an electrostatic potential. The result is

$$B_r = 2\mu_o \left\{ [\cos\theta \cos\theta_m + \sin\theta \sin\theta_m \cos(\varphi - \varphi_m)] \cdot \left[\frac{a \sin(ar-\omega_o t)}{r^2} + \frac{\cos(ar-\omega_o t)}{r^3}\right]\right\}$$

$$B_\theta = -\mu_o \left\{ [\sin\theta \cos\theta_m - \cos\theta \sin\theta_m \cos(\varphi-\varphi_m)] \cdot \left[\frac{a^2 \cos(ar-\omega_o t)}{r} - \frac{a \sin(ar-\omega_o t)}{r^2} - \frac{\cos(ar-\omega_o t)}{r^3}\right]\right\}$$

$$B_\varphi = -\mu_o \sin\theta_m \sin(\varphi-\varphi_m) \left[\frac{a^2\cos(ar-\omega_o t)}{r} - \frac{a \sin(ar-\omega_o t)}{r^2} - \frac{\cos(ar-\omega_o t)}{r^3}\right]. \tag{18}$$

3. Interpretation of the Solution

3.1 Field Distortion: The Longitudinal Electric Field

Equations (16) through (18) specify the electric and magnetic fields which an oscillating magnetic dipole induces in a streaming plasma. The equations hold for arbitrary orientations of $\vec{\mu}_o$ with respect to $\vec{v}_o$, subject to the restriction that $\frac{v_o}{c} << 1$.

Some observations are in order. First [and this is perhaps the most interesting result], it is apparent that an oscillating magnetic dipole, in the presence of a streaming plasma, excites a longitudinal electric field which vanishes when $\vec{v}_o = 0$. The longitudinal normal modes are coupled by the streaming with the transverse electromagnetic modes. Both fields oscillate at the driving frequency of the current source. Second, it is clear that for $\frac{v_o}{c} << 1$ the transverse electric and magnetic fields are essentially undistorted by the streaming plasma. Mathematically, this results from the fact that $\vec{j}_s(\vec{k},\omega)$ for the magnetic dipole is purely transverse. For the electric dipole, on the other hand, $\vec{j}_s(\vec{k},\omega)$ has both a perpendicular <u>and</u> a parallel component. From the general expressions corresponding to (5) and (6) [Bergeson, 1967], we find that the transverse electric field will sustain an order $\frac{v_o}{c}$ distortion when $j_{s_{\parallel}}$ is non-zero. In fact, if we use a current transform appropriate to the oscillating electric dipole, we can recover the fields of Lee and Papas with the present formalism, provided we use Lee and Papas' expression for $\vec{H}$ as a function of $\vec{B}$ and $\vec{E}$. Finally, we note that for $\vec{\mu}_o \parallel \vec{v}_o$ [$\theta_m = \varphi_m = 90°$ in (16) through (18)], there is no longitudinal electric field associated with the oscillating magnetic dipole; an $\vec{E}_{\parallel}$ only exists when $\vec{\mu}_o$ has a non-zero component perpendicular to the streaming velocity $\vec{v}_o$.

3.2 Power Flow: The Skewed Poynting Vector

At great distances from the dipole, only the order $\frac{1}{r}$ terms in the electric and magnetic field equations are significant. We find that the existence -- or perhaps we should say the survival -- of a longitudinal electric field component at large r skews the Poynting vector,

$$\vec{S} = \frac{c}{4\pi} (\vec{E} \times \vec{B}) ,$$

away from the radial orientation that it would have in a purely transverse electromagnetic field. Using (16), (17), and (18) for $\vec{\mu}_o \perp \vec{v}_o$ [$\theta_m = 0°$ in (16) through (18)], we calculate

$$S_r = \frac{\mu_o^2}{4\pi c^3 r^2} \left(\omega_o^4 n_a^3\right) \sin^2\theta \cos^2(ar-\omega_o t)$$

$$S_\theta = 0$$

$$S_\varphi = -\frac{\mu_o^2}{4\pi c^3 r^2} \left(\frac{v_o}{c}\right) (\omega_o n_a)^2 \left(\sum_i \omega_{p_i}^2\right) \sin^2\theta \cos\varphi \cos^2(ar-\omega_o t), \tag{19}$$

$$\text{where } n_a = \left(1 - \frac{\sum_i \omega_{p_i}^2}{\omega_o^2}\right)^{\frac{1}{2}} .$$

The radial component of $\vec{S}$ is the same as it is in a stationary plasma; the azimuthal component of $\vec{S}$ is dependent on $\vec{v}_o$ and points in the "upstream" direction (figure 3). Lee and Papas find a similar result for the electric dipole, i.e. that the Poynting vector is tilted <u>against</u> the direction of zeroth-order plasma flow. [However, they define their Poynting vector in terms of $\vec{E} \times \vec{H}$ rather than $\vec{E} \times \vec{B}$. By using $\vec{B}$, we include only field energy flow in our Poynting flux, rather than include some particle kinetic energy as is done in the "dispersive medium" point-of view.]

3.3 Conservation of Energy: Mechanical Energy Transmission

It can be shown from (1) that a generalization of Poynting's theorem -- applicable to a cold (streaming) plasma -- is

$$\nabla \cdot [\vec{S} + \sum_i (\frac{1}{2} n_i m_i v_i{}^2)\vec{v}_i] + \frac{\partial}{\partial t} [u + \sum_i (\frac{1}{2} n_i m_i v_i{}^2)] = -\vec{E} \cdot \vec{j}_s, \quad (20)$$

where $\vec{S}$ is the Poynting vector, $u = \frac{1}{8\pi}(E^2 + B^2)$ is the electromagnetic energy density and $\vec{j}_s$ is the external source current density. [Field (1956) gives a generalization of Poynting's theorem for a hot (non-streaming) plasma.] We will neglect the time derivative of the energy density in this equation; for sinusoidal oscillations, its time average is zero.

Equation (20) suggests the possibility that mechanical energy transmission may be associated with the longitudinal electric field. In order to examine this idea, we enclose the dipole within a sphere of radius r and form the volume integral of (20) -- without the energy density term. Using the divergence theorem for the integral on the left, we find that

$$\oint_S da\, \hat{n} \cdot [\vec{S} + \sum_i (\tfrac{1}{2} n_i m_i v_i^2)\, \vec{v}_i] = - \int_V d^3x\, (\vec{E} \cdot \vec{J}_s), \qquad (21)$$

where $\hat{n}da$ is an outwardly directed element of surface area and the volume integral on the right is the external power input of the source current. We linearize (21) by setting $\vec{v}_i = \vec{v}_o + \vec{v}_i^{(1)}$ and $n_i = n_{o_i} + n_i^{(1)}$; we may calculate both $\vec{v}_i^{(1)}(\vec{x},t)$ and $n_i^{(1)}(\vec{x},t)$ by using the results of section 2. For $\vec{\mu}_o \perp \vec{v}_o$,

$$\vec{v}_i^{(1)}(\vec{x},t) \simeq -\left(\frac{e_i}{m_i}\right)\left(\frac{\mu_o \omega_o}{c^2\, r}\right) \sin\,(ar-\omega_o t)$$

$$\sin\theta \left\{ \left[\left(\frac{v_o}{c}\right) \cos\varphi \right] \hat{r} + [n_a]\, \hat{\varphi} \right\}$$

$$\simeq \left(v_i{}^{(1)}_r\right) \hat{r} + \left(v_i{}^{(1)}_\varphi\right) \hat{\varphi} \qquad (22)$$

$$n_i^{(1)}(\vec{x},t) \simeq -\left(\frac{n_o e_i}{m_i}\right)\left(\frac{v_o}{c}\right)\left(\frac{\mu_o}{c^2}\right)\left(\frac{\omega_o}{c}\ n_a\right)\ \frac{\sin\theta\cos\varphi}{r}\ \sin(ar-\omega_o t), \tag{23}$$

where we have retained only those terms of order $(\frac{1}{r})$ in distance. When we linearize the mechanical power part of (21) and use (22) and (23) for the first order perturbations, we find that only three terms survive the surface integration and are non-vanishing as r becomes infinitely large. Two of these terms are of order $\frac{v_o^4}{c^4}$ and may be neglected. However, the third term,

$$\frac{1}{2}\ \sum_i \left[2n_{o_i} m_i\left(v_{o_\varphi}\ v_{i_\varphi}^{(1)}\right)\ v_{i_r}^{(1)}\right],$$

is of order $\frac{v_o^2}{c^2}$. As a result, the time-averaged mechanical power flow away from the dipole is

$$\oint\!\!\oint da\left[\sum_i \left(\frac{1}{2}\ n_i m_i v_i^2\right)\ v_{i_r}\right]\Bigg|_{\text{time average}} \simeq$$

$$\frac{1}{6}\left(\sum_i \omega_{p_i}^2\right)\left(\frac{\mu_o^2}{c^2}\right)\left(\frac{\omega_o^2 n_a}{c}\right)\left(\frac{v_o}{c}\right)^2. \tag{24}$$

Therefore, in principle, there is an outward flow of mechanical energy when the dipole oscillates in the presence of a streaming plasma, although the energy transfer is negligible in the limit of small streaming velocities. Since the Poynting flux is accurate only through terms of order $({}^{v}o/c)$, (24) is of no use in verifying the conservation laws.

We compare the outward flux of mechanical energy with the outward flux of electromagnetic energy, as given by the first term in (21). The time-averaged electromagnetic power <u>output</u> is

$$\oiint da\, S_r \Big|_{\text{time average}} = \frac{1}{3} \frac{\mu_o^{\,2}\, \omega_o^{\,4}\, n_a^{\,3}}{c^3} \tag{25}$$

for small $\frac{v_o}{c}$, regardless of whether the plasma is streaming or stationary. It can be shown that performing the volume integration on the right hand side of (21), with the field of (16), also gives (25) for the power <u>input</u>. The additional flux of mechanical power, as given by (24), is of order $(v_o^{\,2}/c^2)$ and is thus negligible.

3.4 Conservation of Momentum: The Mechanical Force on the Dipole

The streaming plasma exerts a mechanical force on the dipole -- a force which tends to push the dipole "downstream", in the direction of flow. Let $\overleftrightarrow{T}$ be the Maxwell stress tensor:

$$\overleftrightarrow{T} = \frac{1}{4\pi}\left[\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2}\overleftrightarrow{I}\,(E^2 + B^2)\right] .$$

We may calculate the force which acts on the dipole by enclosing it again within a spherical boundary surface and evaluating the integral $\oiint \hat{n}\cdot\overleftrightarrow{T}\,da$. Using the complete field components (16), (17), and (18), we find that for $\vec{\mu}_o \perp \vec{v}_o$

$$\oiint_S \hat{n}\cdot\overleftrightarrow{T}\,da\,\Bigg|_{\text{time average}} \simeq \left[\frac{\mu_o^{\,2}\,\omega_o^{\,2}\left(\sum_i \omega_{p_i}^{\,2}\right) n_a}{6c^4}\right]\frac{v_o}{c}\,\hat{\jmath}\,, \qquad (26)$$

where the unit vector $\hat{\jmath}$ points in the direction of flow.

The expression above is equal to the time average of $\frac{d\vec{P}}{dt}$, where $\vec{P}$ is the mechanical momentum of the particles [and the dipole] plus the electromagnetic momentum of the fields within the volume of the sphere. Specifically,

$$\frac{d\vec{P}_{\text{mechanical}}}{dt} = \int_V \left[\rho\vec{E} + \frac{1}{c}\left(\vec{J} \times \vec{B}\right)\right] d^3x ,$$

$$\vec{P}_{\text{field}} = \frac{1}{4\pi c} \int_V \left(\vec{E} \times \vec{B}\right) d^3x .$$

But the time average of the time derivative of $\vec{P}_{\text{field}}$ is zero. Moreover, (26) is independent of r, the radius of the sphere. In the limit as $r \to 0$, the sphere encloses only the dipole $\vec{\mu}_o$ at the origin; it no longer contains any particles of plasma. Hence, we may interpret (26) as the effective mechanical force which the streaming plasma exerts on the dipole. The force is parallel to the direction of flow (figure 4), and it vanishes when either $\vec{v}_o = 0$ or $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}$ is zero. An equal and opposite force must be supplied externally to keep the dipole in place. For $\vec{\mu}_o$ parallel to $\vec{v}_o$, the force on the dipole vanishes. The mechanical work per second necessary to drag the dipole with velocity $\vec{v}_o$ through a quiescent plasma is given by dotting (26) with $\vec{v}_o$. This yields (24), the time-averaged mechanical power flow away from a stationary dipole in a streaming plasma.

There is also a "downstream" force on an oscillating electric dipole immersed in a streaming plasma. However, in the case of the electric dipole, the force does not vanish when the dipole axis is parallel to the direction of the zeroth-order plasma flow.

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Figure Captions

Figure 1. Circular loop of source current immersed in a streaming plasma. The zeroth-order plasma flow vector is $\vec{v}_o$. The unit vectors $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ form a Cartesian triad with $\hat{e}_z$ normal to the plane of the loop; r and φ are the corresponding polar coordinates. The radius of the loop is a_o.

Figure 2. Coordinate configuration for the electric and magnetic fields. The Cartesian coordinate system (xyz) is fixed in space with its y axis directed along $\vec{v}_o$; the xz plane is normal to the direction of plasma flow. The vector $\vec{\mu}_o$ represents the magnetic moment of the point dipole, which is placed at the origin of (xyz). The polar angle θ_m and the azimuthal angle φ_m specify the angular orientation of $\vec{\mu}_o$, while the spherical coordinates r, θ, and φ define the position of an observer.

Figure 3. Poynting vector skew. The dipole is perpendicular to the plane of the paper (the xy plane of figure 2), while the plasma flow is parallel to it. The dark arrows show schematically the direction of the Poynting vector, $\vec{S}$, in the xy plane at various points around a contour of constant $E_\perp^2$. The

angle τ, which specifies the degree of skew, or tilt, of $\vec{S}$ away from the radial direction, is defined by the expression

$$\tan \tau = \frac{|\vec{v}_o|}{c} \left[\frac{\left(\sum_i \omega_{p_i}^{2}\right)}{\omega_o^{2} \, n_a} \right] |\cos \varphi| \, .$$

Figure 4. Contours of constant E^2 and $E_{\parallel}^{\,2}$ (radiation fields). The dipole is perpendicular to the zeroth-order plasma flow, as in figure 3. The ellipse schematically represents a contour of constant E^2, where $|\vec{E}|$ is the magnitude of the total electric field. The lemniscate schematically traces a contour of constant $E_{\parallel}^{\,2}$. The contours lie in the "equatorial" (xy) plane of the dipole. The time-average force which the plasma exerts on the dipole is

$$\vec{F}_{orce} = \left[\frac{\mu_o^{2} \; \omega_o^{2} \left(\sum_i \omega_{p_i}^{2}\right) n_a}{6 \, c^4} \right] \frac{\vec{v}_o}{c}$$

G67-382

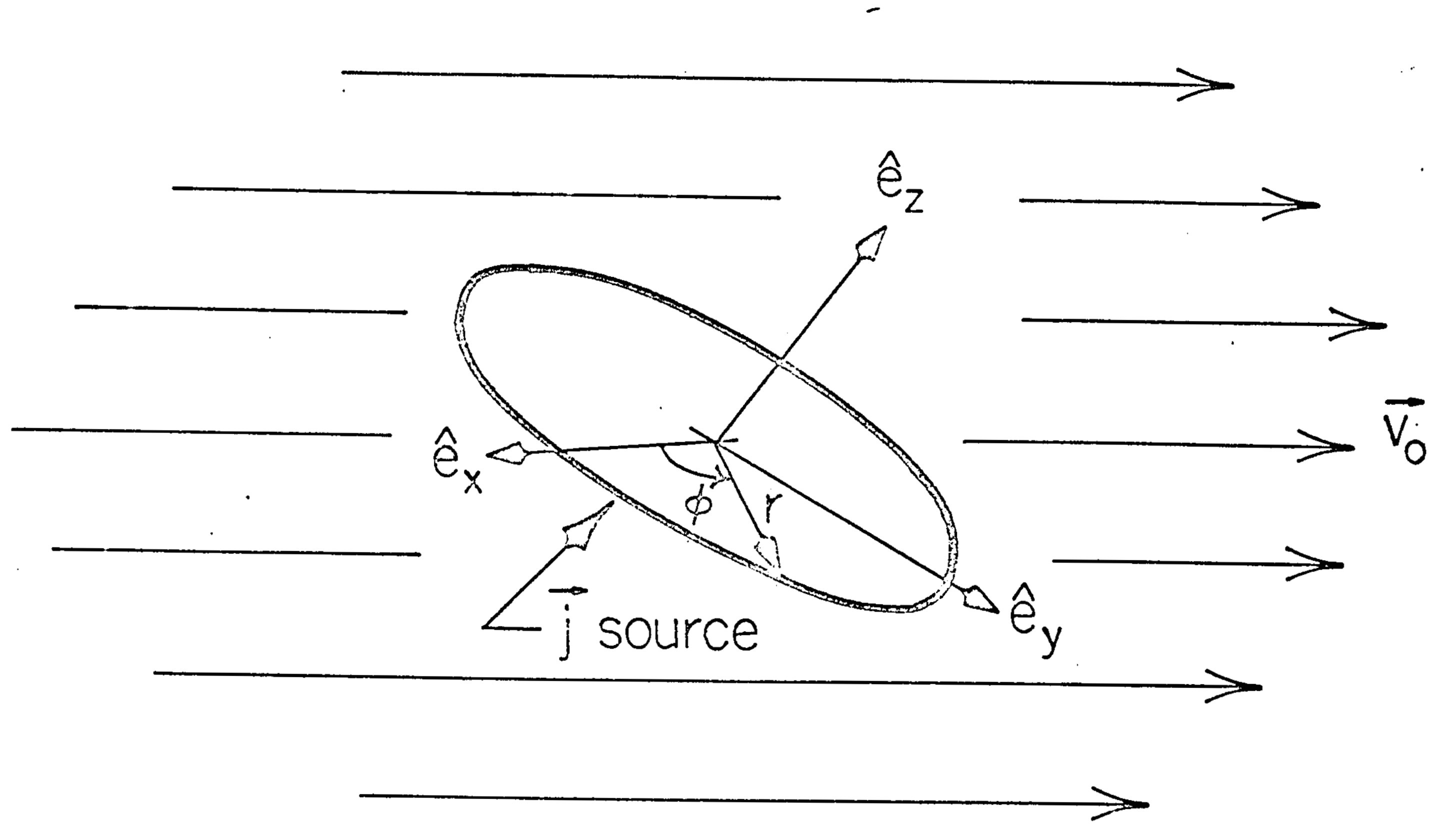

FIGURE 1

G 67-383

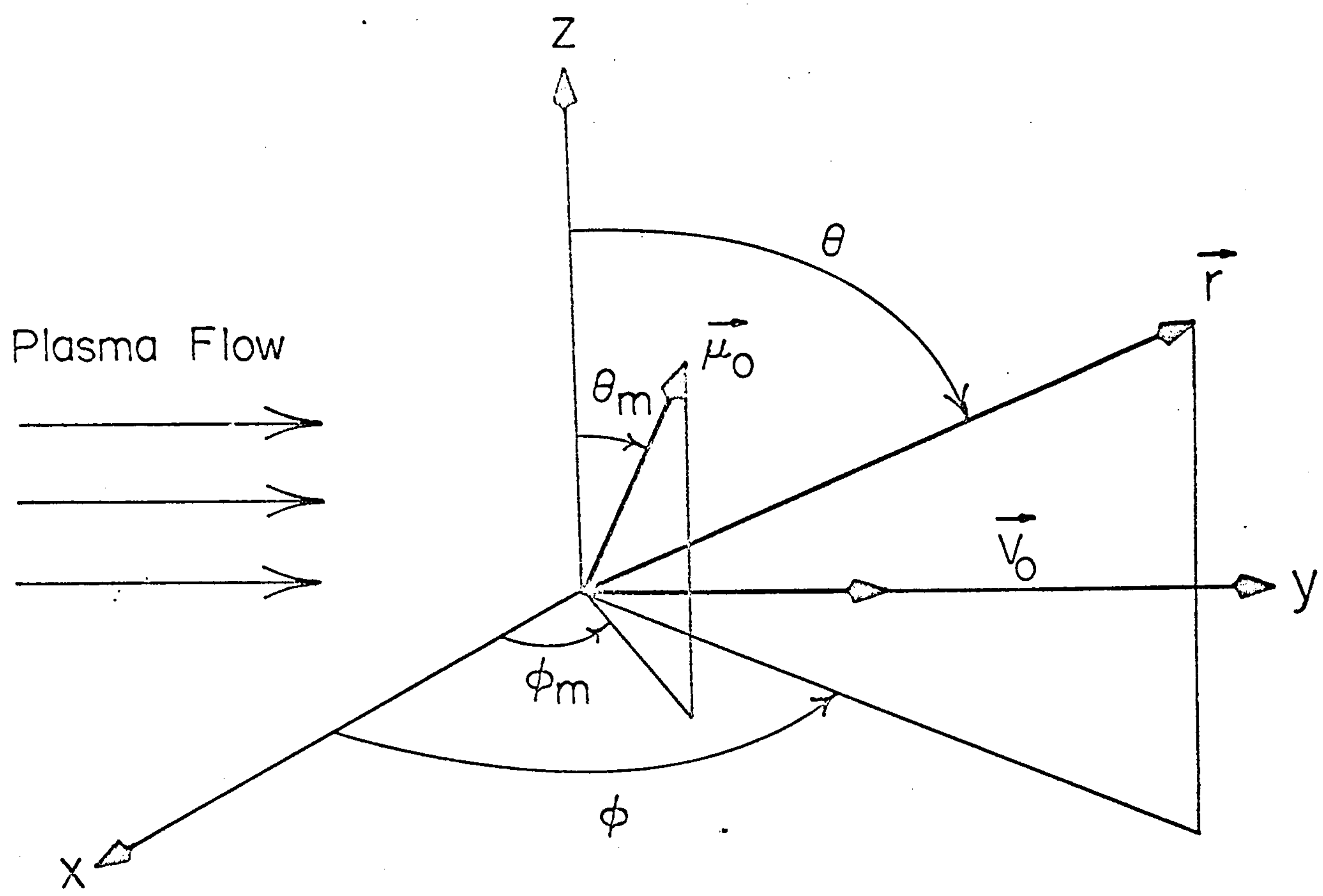

FIGURE 2

G67-384

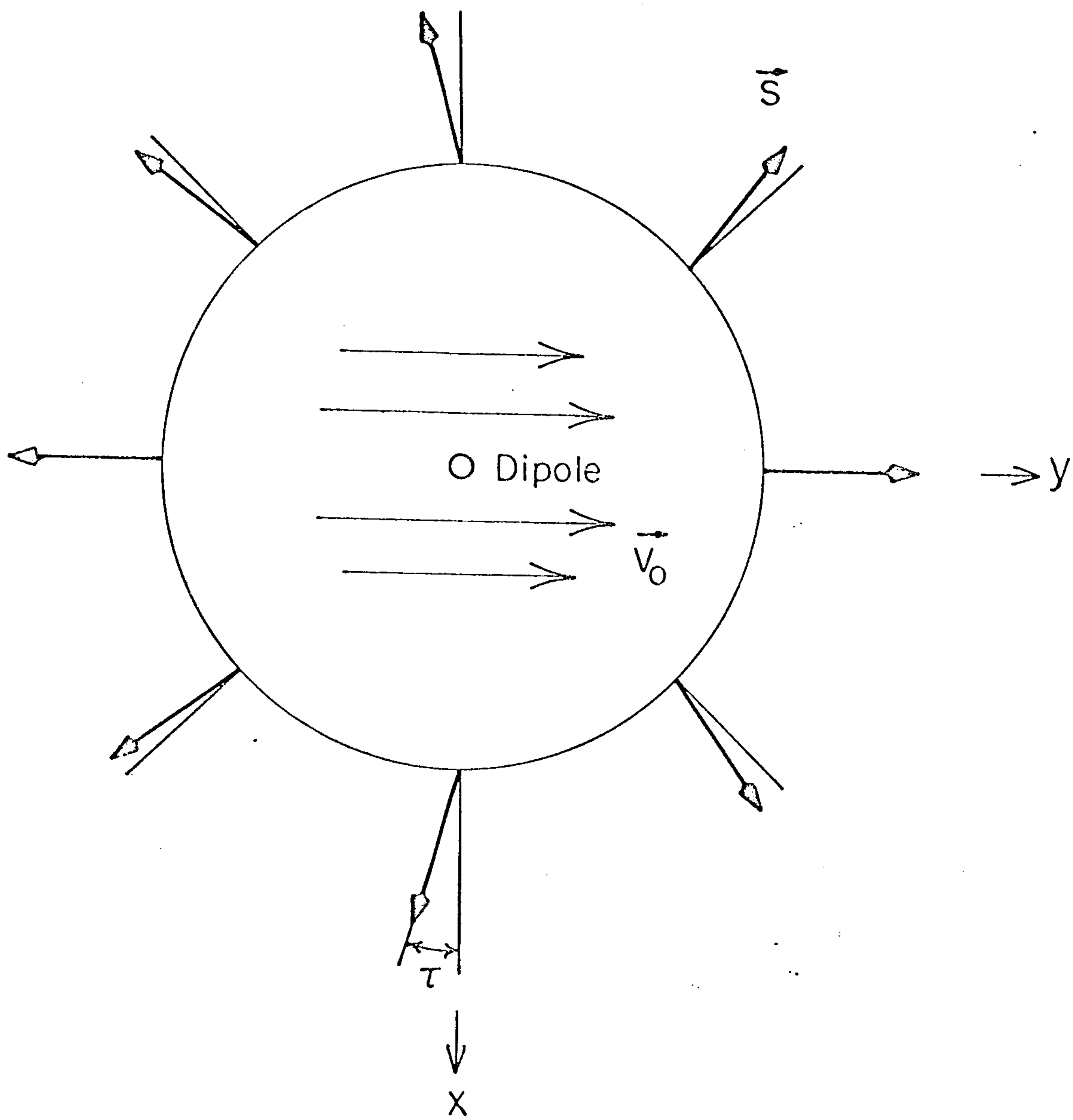

FIGURE 3

G67-385

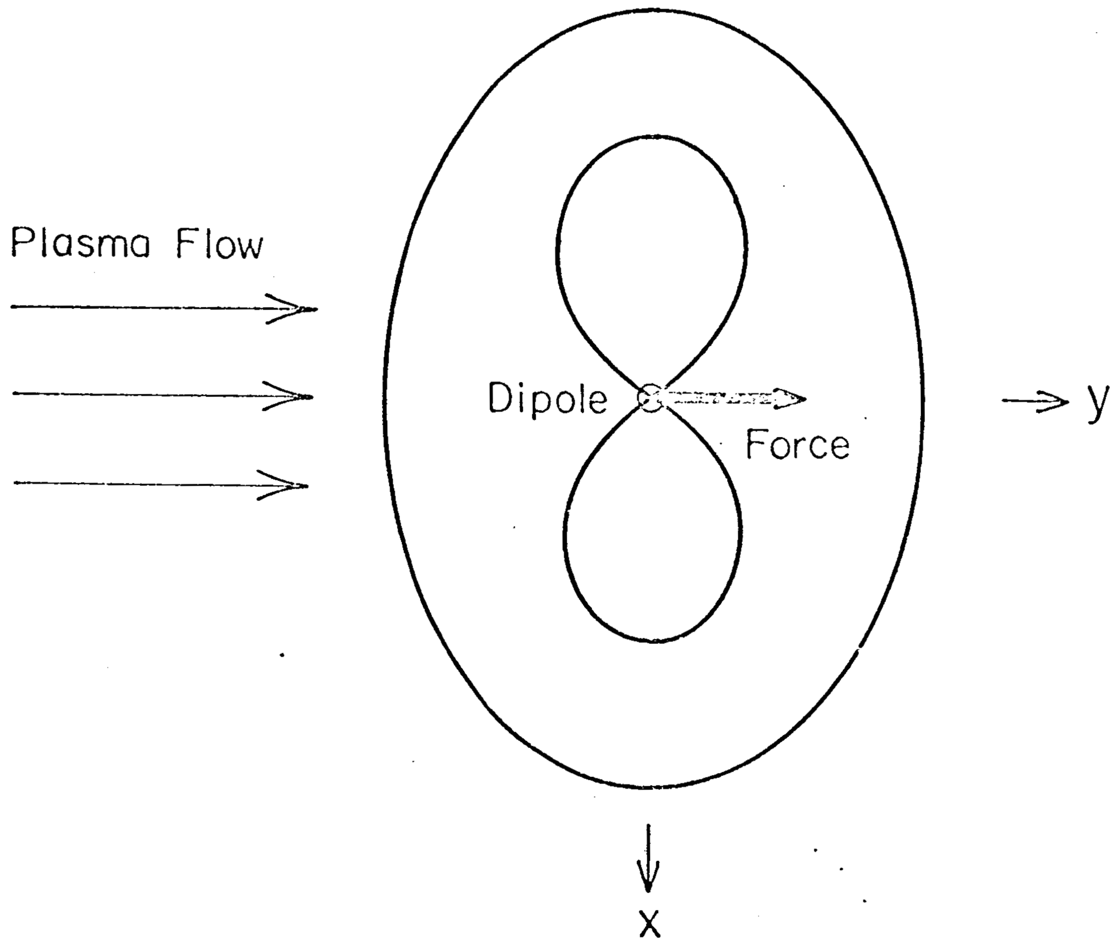

FIGURE 4